

Cosmology in a Higher-Curvature Gravity

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Abstract

We investigated the cosmology in a higher-curvature gravity where the dimensionality of spacetime gives rise to only quantitative difference, contrary to Einstein gravity. We found exponential type solutions for flat isotropic and homogeneous vacuum universe for the case in which the higher-curvature term in the Lagrangian density is quadratic in the scalar curvature, ξR^2 . The solutions are classified according to the sign of the cosmological constant, Λ , and the magnitude of $\Lambda\xi$. For these solutions 3-dimensional space has a specific feature in that the solutions are independent of the higher curvature term. For the universe filled with perfect fluid, numerical solutions are investigated for various values of the parameter ξ . Evolutions of the universes in different dimensionality of spacetime are compared.

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1 Introduction

Einstein gravity describes the observed universe fairly well. It predicts, however, the presence of initial singularity in our universe [1, 2]. This is usually interpreted to mean the limit of applicability of the Einstein gravity. Candidates for the alternative theory come from both classical and quantum considerations. In classical regime, there exists a possibility of avoiding the initial singularity in higher-curvature gravity(HCG) theories. HCG seems to be natural when the gravity is very strong and the curvature is large, since linearity in R of the Lagrangian would be too restrictive. Numerical investigations of the initial singularity in HCG have already been carried out [3] and singularity free solutions have been found, although the solutions are limited only to a very early stage of the universe. HCG is also suggested from the consistency of quantum theory, e.g. quantum field theory in curved spacetimes or string perturbation theory. In these cases tidal effects are important and equivalence principle would not work effectively.

Thus HCG seems to work in both classical and quantum theory. From the cosmological point of view, it may give a better description of the initial stage of the universe than Einstein one and is expected to give a similar description of the later universe as the Einstein one since the curvature effects would then be small. However, the whole evolution of the universe in HCG has not been investigated in detail so far and the description of the present state of

the universe is not satisfactory[4].

There have been many investigations on cosmology in superstring motivated HCG theories. They can be classified into two types. In one of them, the results of the perturbation theory are used and the Gauss-Bonnet combination is adopted.[5] In another, a suitable hypothesis is adopted to construct HCG Lagrangian density leading to produce non-singular cosmological solutions, instead of using the results of the perturbation theory. [6, 7, 8, 9] Both types of models yield interesting cosmological solutions, although they are not satisfactory, since the solutions do not describe the present state of the universe.

Theoretically the Gauss-Bonnet combination has a nicely simple property. However it is so simple that new generalized coordinates, coming from the time derivatives of the metric and inherent to HCG, do not appear in the canonical formalism at least for the Robertson-Walker spacetime. Therefore it would be worthwhile to investigate other types of HCG. One of the simple models is described by a function of the scalar curvature. This type of models has long been investigated and includes many problems peculiar to HCG such as the definition of the new generalized coordinate, complicated constraint structure[10, 12, 13] and conformal equivalence[11]. Thus it is interesting to see whether this type of models are applicable to describe the realistic evolution of the universe.

In this paper, we investigate the evolution of the universe in HCG, described by a function of the scalar curvature, using analytical and numerical methods. In numerical analyses, we adopt the recent observational data as the initial data which leads to the accelerated expansion, instead of assuming suitable initial data at $t = 0$. One of the main differences of HCG from Einstein one is that the dimension of the spacetime gives rise to only quantitative differences, e.g. while in Einstein gravity there is no dynamical degrees of freedom in (2+1)-dimensional spacetime, there remains such freedom in HCG [12, 13]. Therefore it is easy to compare spacetimes differing in dimensionality[14]. In other words it would be easy to transfer results obtained for lower dimensional spacetime to higher dimensional one. We investigated the universe for the case of vacuum universe and the universe filled with perfect fluid for various values of the coefficient ξ of the R^2 -term. It is found that (3+1)-dimensional spacetime has somewhat peculiar property in the former case.

In section 2 we present the basic equations for general spacetimes as well as for homogeneous and isotropic universe in the case of R^2 gravity. In section 3 analytical and numerical solutions are investigated. We obtain exponentially evolving solutions for a particular model containing a term quadratic in scalar curvature. The results of numerical investigations for the vacuum case and the universe filled with perfect fluid are presented for various values of ξ . The solutions exhibit largely different behavior according to the sign of ξ . Section 4 is devoted to the summary and discussions.

2 Formulation

2.1 Basic equations

We consider a pure gravity type model in $(d + 1)$ -dimensional spacetime described by a Lagrangian density

$$\mathcal{L}_G = \frac{1}{16\pi\hat{G}}\sqrt{-\hat{g}}f(R) \quad (2.1)$$

where f is a differentiable function of the $(d + 1)$ -dimensional scalar curvature R and a hat denotes a quantity defined in $(d + 1)$ -dimensional spacetime. Field equations derived from

(2.1) is known to be written as

$$f' R_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} f + \hat{g}_{\mu\nu} \square f' - \nabla_\mu \nabla_\nu f' = 0 \quad (2.2)$$

where $R_{\mu\nu}$ is the Ricci tensor constructed from the $(d+1)$ -dimensional metric $\hat{g}_{\mu\nu}$, ∇_μ is the covariant derivative with respect to $\hat{g}_{\mu\nu}$, $\square \equiv \hat{g}^{\lambda\rho} \nabla_\lambda \nabla_\rho$ and $f' \equiv df/dR$.

Equations (2.2) depend on the dimension of spacetime only implicitly. The dependence becomes explicit in the canonical formalism. The system described by the Lagrangian density (2.1) is highly constrained, so we follow the method of Buchbinder and Lyakhovich (BL) [10, 12] for treating the higher-derivative Lagrangian. The Hamiltonian density \mathcal{H}^* is obtained by the Legendre transformation of the modified Lagrangian density \mathcal{L}_G^* which takes into account the constraints and the definitions of the generalized coordinates coming from the time derivatives of the metric by introducing the Lagrange multipliers. \mathcal{H}^* has the following form:

$$\mathcal{H}_G^* = N\mathcal{H}_G + N^k \mathcal{H}_k + \text{divergent terms} \quad (2.3)$$

where N is the lapse function and N^k is the shift vector. $\mathcal{H}_G = 0$ and $\mathcal{H}_k = 0$ are the Hamiltonian and momentum constraints, respectively. Choosing the coordinate system where the lapse function N is equal to 1 and the shift vector N^k vanishes, explicit form for $\mathcal{H}_G = \mathcal{H}_G^*$ is given as follows[12]:

$$\mathcal{H}_G = 2\Pi^{-1} \left(p^{ij} p_{ij} - \frac{1}{d} p^2 \right) + \frac{2}{d} p Q + \frac{1}{2} \Pi R - \sqrt{h} f(R) - \frac{d+1}{2d} \Pi Q^2 + \Delta \Pi - \frac{1}{2} \Pi \mathcal{R} \quad (2.4)$$

where h_{ij} is the metric of the d -dimensional space, p^{ij} the momentum canonically conjugate to h_{ij} , $h \equiv \det h_{ij}$, \mathcal{R} the scalar curvature formed from h_{ij} . $p \equiv h_{ij} p^{ij}$, $Q \equiv h^{ij} Q_{ij}$ where Q_{ij} is the extrinsic curvature which is taken as new generalized coordinate replacing the time-derivative of h_{ij} . Π is the momentum canonically conjugate to Q . The scalar curvature of the $(d+1)$ -dimensional spacetime, R , should be expressed in terms of the canonical variables through a relation $\Pi = 2\sqrt{h} f'(R)$. It is seen that the dimension d appears only in the coefficients of some of the terms in such a way that there is no special value of d .

2.2 Homogeneous, isotropic and flat spacetimes

In order to solve the field equations explicitly, we have to specify the model. Here we specialize the function $f(R)$ to a quadratic function:

$$f(R) = -2\Lambda + R + \xi R^2 \quad (2.5)$$

Then, in terms of canonical variables, R is expressed as

$$R = \frac{1}{2\xi} \left(\kappa^2 \Pi / \sqrt{h} - 1 \right) \quad (2.6)$$

where $\kappa = 8\pi G$. Furthermore we specialize the spacetime to the flat Robertson-Walker spacetime:

$$ds^2 = -dt^2 + a(t)^2 \tilde{h}_{ij} dx^i dx^j \quad (2.7)$$

where \tilde{h}_{ij} is the metric of flat d -dimensional static space. Then \mathcal{H}_G in (2.4) reduces to the following form:

$$\mathcal{H}_G = \frac{\kappa}{8\xi} a^{-2} \Pi^2 - \frac{1}{4} \left(3Q^2 + \frac{1}{\xi} \right) \Pi + \frac{1}{2} p_a a Q + \frac{1}{\kappa} \left(\Lambda + \frac{1}{8\xi} \right) a^2 \quad (2.8)$$

where p_a is the momentum canonically conjugate to the scale factor a .

For vacuum spacetime the canonical equations of motion are derived from (2.8):

$$\begin{cases} \dot{a} = \frac{1}{d}aQ, & \dot{p}_a = \frac{d\kappa}{8\xi}a^{-(d+1)}\Pi^2 - \frac{1}{d}Qp_a - \frac{d}{\kappa}\left(\Lambda + \frac{1}{8\xi}\right)a^{d-1}, \\ \dot{Q} = \frac{\kappa}{4\xi}a^{-d}\Pi - \frac{1}{2}\left(\frac{d+1}{d}Q^2 + \frac{1}{\xi}\right), & \dot{\Pi} = \frac{1}{d}\left\{(d+1)Q\Pi - ap_a\right\} \end{cases} \quad (2.9)$$

From these equations, we have an equation for a ¹:

$$(d-1)\frac{\ddot{a}}{a} + \frac{1}{2}(d-1)(d-2)\left(\frac{\dot{a}}{a}\right)^2 - \Lambda + HCT = 0 \quad (2.10)$$

where HCT represents the contribution of the higher curvature terms and is expressed as

$$\begin{aligned} HCT = & d\xi \left[4\left(\frac{a^{(4)}}{a}\right) + 8(d-2)\left(\frac{\dot{a}}{a}\right)\left(\frac{a^{(3)}}{a}\right) + 6(d-2)\left(\frac{\ddot{a}}{a}\right)^2 \right. \\ & \left. + 2(3d^2 - 20d + 21)\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\ddot{a}}{a}\right) + \frac{1}{2}(d-1)(d-4)(d-9)\left(\frac{\dot{a}}{a}\right)^4 \right]. \end{aligned} \quad (2.11)$$

3 Solutions

3.1 Analytical solutions

It is easily seen that (2.10) allows a particular solution of the form $a(t) = a_0 e^{\lambda t}$ with constant a_0 . Putting this form into (2.10), we have

$$d(d-1)\lambda^2 - 2\Lambda + \xi d^2(d+1)(d-3)\lambda^4 = 0 \quad (3.1a)$$

so that

$$\lambda^2 = \left[-d(d-1) \pm \sqrt{d^2(d-1)^2 + 8\Lambda\xi d^2(d+1)(d-3)} \right] / 2\xi d^2(d+1)(d-3) \quad (3.1b)$$

Thus the solutions are classified according to the value of

$$\eta \equiv 8\Lambda\xi(d+1)(d-3)/(d-1)^2, \quad (3.2)$$

and the sign of Λ . The solutions exist only for $\eta \geq -1$.

$$\begin{aligned} \text{I. } a(t) &= a_0 \exp \left[\pm \sqrt{\frac{4\Lambda}{d(d-1)}} t \right] & \text{for } \eta = -1 & \quad \text{and } \Lambda > 0 \\ \text{II. } a(t) &= a_0 \exp \left[\pm \sqrt{\frac{4\Lambda}{d(d-1)\eta}} \left(-1 \pm \sqrt{1+\eta} \right) t \right] & \text{for } -1 < \eta < 0 & \quad \text{and } \Lambda > 0 \\ \text{III. } a(t) &= a_0 \exp \left[\pm \sqrt{\frac{4\Lambda}{d(d-1)\eta}} \left(-1 + \sqrt{1+\eta} \right) t \right] & \text{for } \eta > 0 & \quad \text{and } \Lambda > 0 \\ a(t) &= a_0 \exp \left[\pm \sqrt{\frac{4|\Lambda|}{d(d-1)\eta}} \left(1 + \sqrt{1+\eta} \right) t \right] & \text{for } \eta > 0 & \quad \text{and } \Lambda < 0 \end{aligned} \quad (3.3)$$

¹Of course, (2.10) can be derived from the space components of (2.2).

This classification can be made more directly (although not compact), e.g. $\eta = -1$ means $\Lambda\xi = -(d-1)^2/8(d+1)(d-3)$ or $\eta > 0$ means $\Lambda\xi(d-3) > 0$. It should be noted that, for $d = 3$, (3.1a) shows that there is no contribution from the higher curvature term, so the solution is the same as in Einstein gravity. This is also seen if the trace of (2.2) is taken and (2.5) is used for $f(R)$. Correspondingly, for $d = 3$, and consequently $\eta = 0$, only half of the type II and type III solutions have $\eta \rightarrow 0$ limit. We note that, for $\Lambda = 0$, nontrivial solutions exist only for $d = 2$ if $\xi > 0$. It would be expected that these exponential type vacuum solutions are relevant only to the early stage of the universe. Thus, in order to examine whether the initial singularity of our universe can be avoided or not, other types of solutions are necessary. They will be investigated numerically in the next subsection.

In later stage of the universe, we assume the universe to be filled with perfect fluid, when the field equations are obtained by replacing the rhs of (2.2) by the energy-momentum tensor of the perfect fluid. Equation for the scale factor a is expressed as

$$(d-1)\frac{\ddot{a}}{a} + \frac{1}{2}(d-1)(d-2)\left(\frac{\dot{a}}{a}\right)^2 - \Lambda + HCT = \kappa p. \quad (3.4)$$

The field equations are supplemented by the equation of state

$$p = \gamma\rho \quad (3.5)$$

where p is the pressure, ρ the total energy density and γ a constant. Energy-momentum conservation leads to a relation:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{d(\gamma+1)}. \quad (3.6)$$

Analytical solutions are available only for radiation dominated era, $\gamma = 1/d$, which reads

$$a(t) \propto t^{2/(d+1)}. \quad (3.7)$$

Also in this case, $d = 3$ is exceptional in that the solution (3.7) is the same as in Einstein gravity. Other solutions are investigated numerically in the next section.

3.2 Numerical analysis

In this section we present the results of the numerical analyses of the equation (2.10), for the vacuum case, and (3.4) with (3.5), describing the universe filled with perfect fluid. These equations were solved numerically by the 4th order Runge-Kutta method.

We start from the universe today and investigate the evolution of the universe both to the future and to the past. The present values of the cosmological parameters are taken from the WMAP results [15]. The energy densities of matter ρ_{M0} and dark energy (cosmological constant) ρ_{DE0} are taken to satisfy

$$(i) \quad \rho_{M0} : \rho_{DE0} = 27 : 73, \quad \text{and} \quad (ii) \quad \Omega_{M0} + \Omega_{DE0} = 1 \quad (\text{flat universe}). \quad (3.8)$$

The latter means, of course, the total energy density is critical. The Hubble constant is taken to be

$$H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (3.9)$$

The value of \ddot{a}_0 is determined approximately from the Einstein equation and $a_0^{(3)}$ from the Hamiltonian constraint. It is noted that these values of parameters lead to the accelerating universe as is shown in the following figures.

Figure 1 shows the predicted evolutions of the universe filled with perfect fluid for $d = 3$ according to Einstein gravity and HCG for various values of the parameter ξ . For positive ξ , the evolutions predicted by HCG are almost the same as that by Einstein gravity as expected. On the other hand, for negative ξ , predictions of HCG are very different from that of Einstein gravity and depend heavily on the values of ξ . These results remain unchanged for the value of H_0 other than the one in (3.9).²

Figure 2 shows the evolutions of the scalar curvature for the cases as in Figure 1. The curvature becomes very large at early stage as expected and corresponds to the behavior of the scale factor. The results shown in Figures 1 and 2 seem to rule out the cases of negative ξ since the calculated age of the universe is so short.

Figure 3 shows the evolution of the scale factor for the dimensions of the space, $d = 2, 3, 4$ in Einstein gravity. As to the initial conditions for $d = 2$ and $d = 4$, we tentatively adopted the same ones as for $d = 3$. Figure 4 shows the corresponding evolution of the scalar curvature.

Figures 5 and 6 show the same evolution as in Figures 3 and 4 in HCG for $\xi = 10$. The difference in the equation of motion is only quantitative as noted above, which is in accordance with our numerical results. We note that our numerical solutions satisfy the Hamiltonian constraint fairly well.

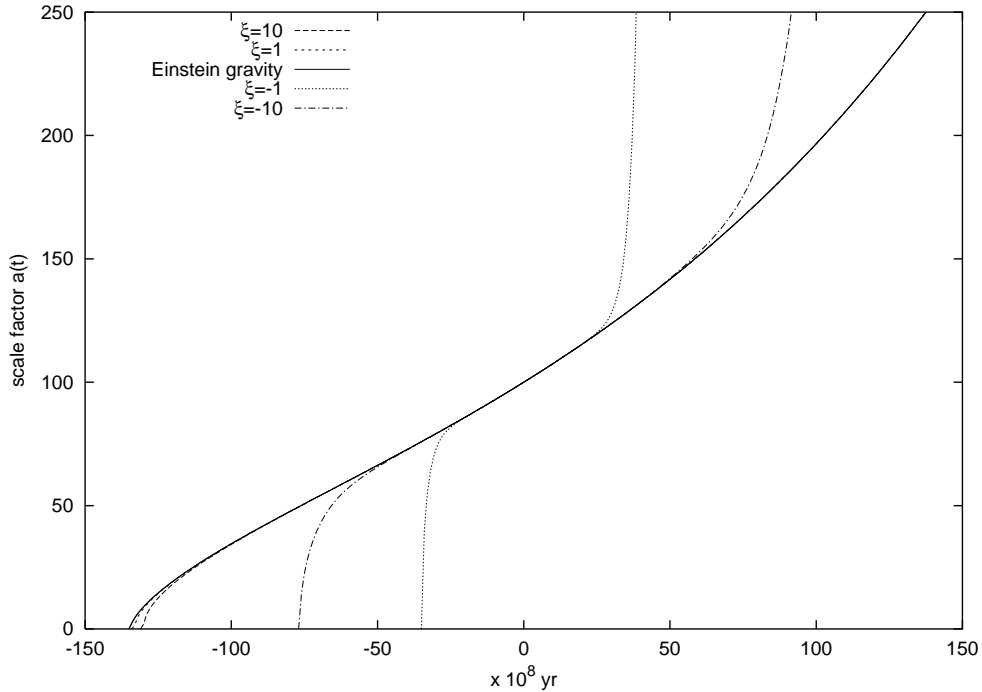


Fig.1 Evolutions of the (3+1)-dimensional universe filled with perfect fluid in Einstein gravity and HCG for various values of ξ

²The older values of these parameters[16, 17, 18, 19, 20] lead to almost the same results.

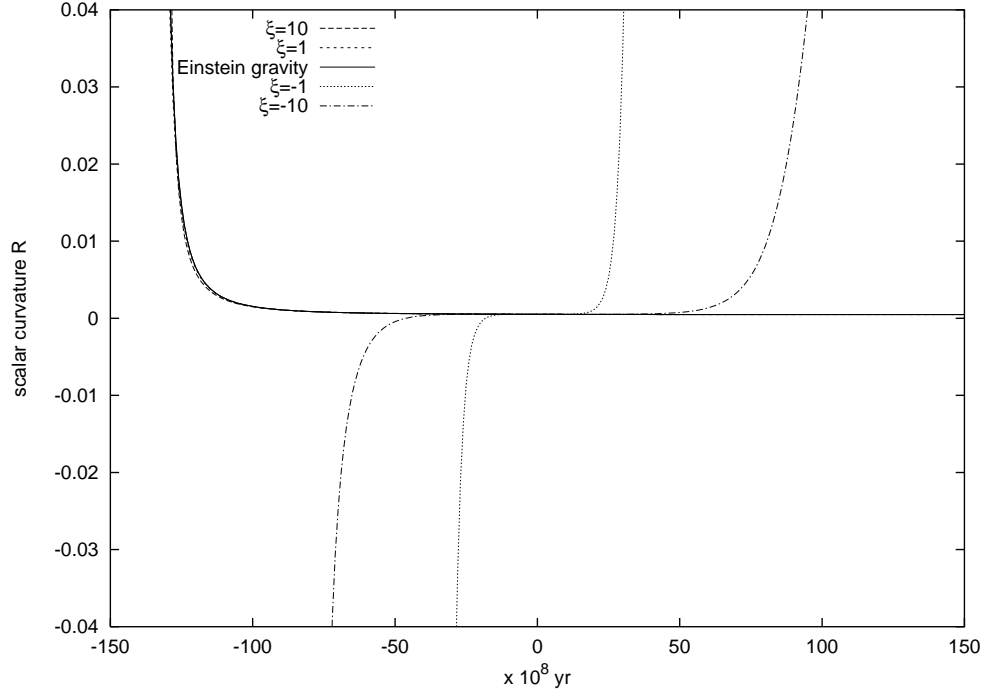


Fig.2 Evolutions of scalar curvature R in (3+1)-dimensional spacetime

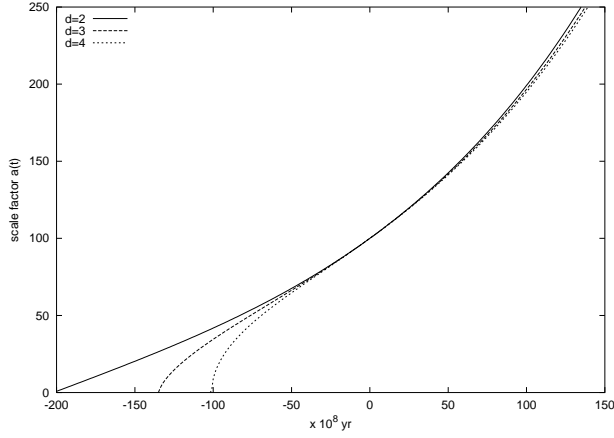


Fig.3 Evolutions of the universe in Einstein gravity for dimensions $d = 2, 3$ and 4.

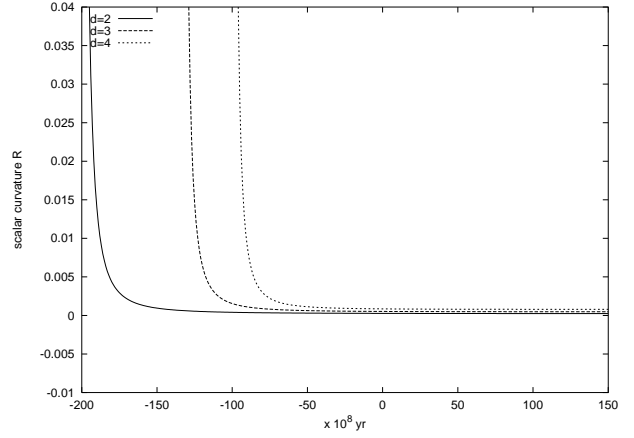


Fig.4 Evolutions of scalar curvature R in Einstein gravity for dimensions $d = 2, 3$ and 4.

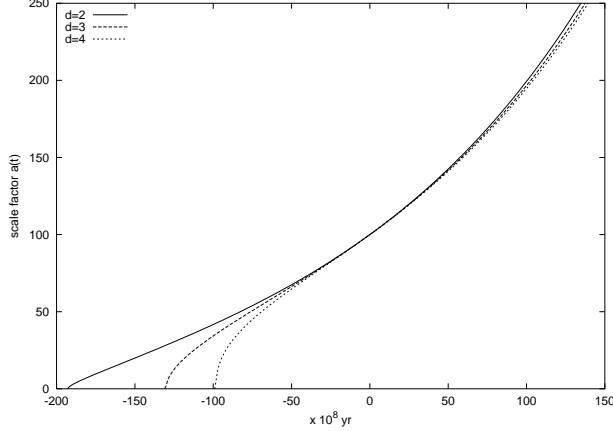


Fig.5 Evolution of the universe in HCG with $\xi = 10$, for dimensions $d = 2, 3$ and 4

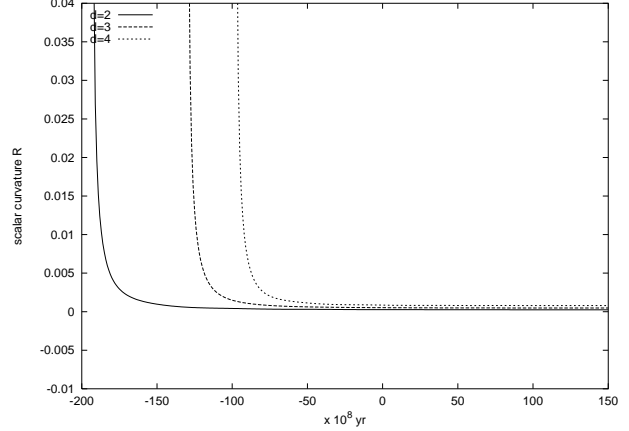


Fig.6 Evolutions of scalar curvature R in HCG for $\xi = 10$ for dimensions $d = 2, 3$ and 4

4 Summary and discussion

We investigated the evolution of the flat, RW universe in a higher curvature gravity (HCG) theory described by a Lagrangian depending on the scalar curvature. Dimensionality of spacetime affects the evolution only quantitatively. Explicit solutions are obtained when the higher curvature term is quadratic in the scalar curvature, ξR^2 . Solutions exhibiting exponential expansion are obtained for the vacuum universe. Exceptional case is that of 3-dimensional space, in which the solution is the same as in Einstein gravity, i.e. the higher curvature term gives rise to no effect. These solutions may be interpreted to mean that other type of solution is necessary to avoid the initial singularity since the universe has not expanded exponentially throughout its evolution.

Numerical investigations were made by taking the recent observational data as initial conditions. The solutions were compared for various values of the parameter ξ and Einstein gravity. For positive ξ , solutions in HCG deviate only slightly from that in Einstein gravity. For negative ξ , solutions depend heavily on the value of ξ and indicate too short an age of the universe. However, numerical results do not show indications of avoiding the initial singularity in the range where our numerical calculations are valid. It may be that a single set of equations is not sufficient to describe the whole history of the universe and different set of equations are required for the earliest and later stages of the universe analogous to the usual inflationary scenario. In other words, it may be required that numerical solutions should be connected to solutions, e.g. those obtained in the previous section at some early time.

Concerning the initial singularity, it is known that for spacetime with physically reasonable properties, occurrence of the singularity is inevitable if the Ricci tensor satisfies $R_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$ for any timelike vector ζ^μ . In HCG with ξR^2 term, we have

$$R_{\mu\nu}\zeta^\mu\zeta^\nu = \frac{1}{1+2\xi R} \left[\kappa \left(T_{\mu\nu}\zeta^\mu\zeta^\nu + \frac{1}{d-1}T \right) - \frac{2}{d-1}\Lambda \right] - \frac{\xi}{(d-1)(1+2\xi R)} \left[R^2 + 2\Box R - 2(d-1)\zeta^\mu\zeta^\nu\nabla_\mu\nabla_\nu R \right]. \quad (4.1)$$

The second term on the rhs is due to the higher curvature effect. It is difficult to determine the sign of the rhs generally, even if the strong energy condition is satisfied. It is seen that

the results may change according the sign of ξ . Restricting to the isotropic and flat universe filled with pressureless perfect fluid, we have, in terms of the canonical variables

$$R_{\mu\nu}\zeta^\mu\zeta^\nu = \frac{\sqrt{h}}{\kappa\Pi} \left[(u_\mu\zeta^\mu)^2 - \frac{1}{d-1} \right] \rho - \frac{2}{d-1}\Lambda - \frac{1}{2(d-1)\xi} \left[\frac{\kappa^2\Pi}{\sqrt{h}} + \frac{\sqrt{h}}{\kappa^2\Pi} - 2 \right] - \frac{\square\Pi}{(d-1)\Pi} + \frac{\zeta^\mu\zeta^\nu\nabla_\mu\nabla_\nu\Pi}{\Pi}.$$

Even in this case the situation is hardly improved so that numerical analysis seems to be necessary. Our numerical results show that R takes large positive value for $\xi > 0$. This indicates the possibility that the second term on the rhs of (4.1) dominates the first term and the numerical solutions could be connected to some non-singular solution.

We also compared the evolution of the universe for the dimensions of space $d = 2, 3$ and 4 . Intuitively the case of lower dimensional spacetime would seem to be simpler [14]. However we found that the dimension of the spacetime is almost irrelevant in HCG.

Finally we comment on the conformal transformations in HCG[11]. In this work we identified the metric $\hat{g}_{\mu\nu}$ in (2.1) as physical and applied the initial conditions to them. If the transformed metric is identified to be physical, the initial conditions are changed, although the equations of motion are only transformed. The transformation is suggested by the form of \mathcal{H}_G in (2.4) which is so complex. The form can be brought to a simpler form. Investigation of the transformed equations would be interesting since new types of solutions could be obtained.

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